

A Hopf theorem for non-constant mean curvature and a conjecture of A.D. Alexandrov

José A. Gálvez and Pablo Mira

ABSTRACT. We prove a uniqueness theorem for immersed spheres of prescribed (non-constant) mean curvature in homogeneous three-manifolds. In particular, this uniqueness theorem proves a conjecture by A.D. Alexandrov about immersed spheres of prescribed Weingarten curvature in \mathbb{R}^3 for the special but important case of prescribed mean curvature. As a consequence, we extend the classical Hopf uniqueness theorem for constant mean curvature spheres to the case of immersed spheres of prescribed antipodally symmetric mean curvature in \mathbb{R}^3 .

1. Introduction

In his famous 1956 paper, A.D. Alexandrov [A2] conjectured the following result:

CONJECTURE 1.1. *Let $S \subset \mathbb{R}^3$ be a strictly convex sphere, and let $\Phi(k_1, k_2, x) \in C^1(\Omega)$ be a function such that*

$$\frac{\partial \Phi}{\partial k_1} \frac{\partial \Phi}{\partial k_2} > 0$$

on the domain $\Omega \subset \mathbb{R}^2 \times \mathbb{S}^2$ given by

$$\Omega = \{(\lambda \kappa_1(p), \lambda \kappa_2(p), \nu(p)) \in \mathbb{R}^2 \times \mathbb{S}^2 : p \in S, \lambda \in \mathbb{R}\},$$

where $\nu : S \rightarrow \mathbb{S}^2$ is the Gauss map of S , and $\kappa_1, \kappa_2 : S \rightarrow \mathbb{R}$ are its principal curvatures.

Let $f : \mathbb{S}^2 \rightarrow \mathbb{R}$ be the function defined by

$$(1.1) \quad \Phi(\kappa_1(p), \kappa_2(p), \nu(p)) = f(\nu(p)) \quad \forall p \in S.$$

Then any other compact surface Σ of genus zero immersed in \mathbb{R}^3 whose Gauss map ν and principal curvatures κ_1, κ_2 satisfy (1.1) is a translation of S .

This conjecture is known to hold when Σ is also a strictly convex sphere ([A1, A2, P1, HW]; see [GWZ] for a historical account of the problem and a generalization). In particular, this provides uniqueness for geometric problems formulated in terms of the *curvature radii* $1/\kappa_i$ such as the Christoffel-Minkowski problem in \mathbb{R}^3 , as the solutions to these problems are automatically strictly convex. Alexandrov stated without proof in [A2] that the conjecture holds provided both Σ, S are real analytic.

In this paper we prove the Alexandrov conjecture above in the special but important case of prescribed mean curvature spheres, i.e. when $\Phi = k_1 + k_2$. Observe that in this situation the immersed compact surface Σ is not assumed to be strictly convex. In order to state our results, we fix some notation. From now on, all surfaces will be assumed to be (at least) of class C^3 .

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DEFINITION 1.2. Given $\mathcal{H} \in C^1(\mathbb{S}^2)$, an immersed oriented surface Σ in \mathbb{R}^3 with Gauss map $\nu : \Sigma \rightarrow \mathbb{S}^2$ is said to have *prescribed mean curvature* \mathcal{H} if

$$H_\Sigma(p) = \mathcal{H}(\nu(p))$$

for every $p \in \Sigma$, where H_Σ is the mean curvature of Σ .

THEOREM 1.3. *Let $S \subset \mathbb{R}^3$ be a strictly convex sphere with prescribed mean curvature $\mathcal{H} \in C^1(\mathbb{S}^2)$. Then S is (up to translations) the only immersed compact surface of genus zero in \mathbb{R}^3 with prescribed mean curvature \mathcal{H} .*

We will actually prove Theorem 1.3 as a particular case of a more general uniqueness theorem for immersed spheres of prescribed mean curvature in simply connected Riemannian homogeneous three-manifolds not isometric to the product space $\mathbb{S}^2(\kappa) \times \mathbb{R}$ (see Theorem 4.1). In particular, Theorem 4.1 covers the situation of prescribed mean curvature spheres in \mathbb{H}^3 or \mathbb{S}^3 , where no similar result seems to be known.

REMARK 1.4. *A classical result by Bonnet states that if there exists a diffeomorphism $\Psi : S_1 \rightarrow S_2$ between two compact immersed surfaces S_1, S_2 of genus zero in \mathbb{R}^3 such that Ψ preserves both the metric and the mean curvature function of the surfaces, then S_1 and S_2 are congruent in \mathbb{R}^3 . Note that in Theorem 1.3 we are not assuming that the spheres S and Σ are isometric, which is a key hypothesis of Bonnet's problem.*

A famous theorem by H. Hopf (see for instance [Ho]) asserts that any compact constant mean curvature surface of genus zero immersed in \mathbb{R}^3 is a round sphere. As a consequence of Theorem 1.3 we obtain a generalization of Hopf's theorem to the case of prescribed (not necessarily constant) antipodally symmetric mean curvature, as we explain next.

In their seminal paper [GG], B. Guan and P. Guan proved that if $\mathcal{H} \in C^2(\mathbb{S}^2)$ satisfies $\mathcal{H}(-x) = \mathcal{H}(x) > 0$ for all $x \in \mathbb{S}^2$, then there exists a closed strictly convex sphere $S_\mathcal{H}$ in \mathbb{R}^3 with prescribed mean curvature \mathcal{H} . Note that when \mathcal{H} is constant, $S_\mathcal{H}$ is a round sphere of radius $1/\mathcal{H}$. Thus, the next corollary is a wide generalization of Hopf's theorem to the case of non-constant mean curvature:

COROLLARY 1.5. *Let Σ be an immersed compact surface of genus zero in \mathbb{R}^3 with prescribed mean curvature $\mathcal{H} \in C^2(\mathbb{S}^2)$, where $\mathcal{H}(-x) = \mathcal{H}(x) > 0$. Then Σ is the Guan-Guan sphere $S_\mathcal{H}$ (up to translation).*

Corollary 1.5 follows directly from Theorem 1.3 and the existence of the Guan-Guan strictly convex spheres in [GG] mentioned above.

We have organized the paper as follows. In Section 2 we give some basic preliminaries about the geometry of simply connected homogeneous three-manifolds X not isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$, and explain how they admit an underlying Lie group structure. That is, they can be seen as *metric Lie groups*. In Section 3 we consider conformally immersed surfaces $\psi : \Sigma \rightarrow X$ in metric Lie groups, and deduce an equation that links the Gauss map and the mean curvature of ψ . This can be seen as a Weierstrass type representation for surfaces in metric Lie groups with given mean curvature and Gauss map, in the spirit of the classical Kenmotsu formula [Ke].

In Section 4 we prove our main uniqueness result (Theorem 4.1) about prescribed mean curvature spheres in metric Lie groups X . Specifically, we prove that the existence of a compact surface S of genus zero in X with prescribed mean curvature $\mathcal{H} \in C^1(\mathbb{S}^2)$ and whose Gauss map is a diffeomorphism to \mathbb{S}^2 implies the existence of a (non-holomorphic) complex quadratic differential for surfaces of prescribed mean curvature \mathcal{H} in X , which vanishes identically on open pieces of S and only has isolated zeros of negative index for any other surface. Theorem 4.1 follows then from the existence of this Hopf type differential by the Poincaré-Hopf theorem, and reduces to Theorem 1.3 when the homogeneous manifold X is the Euclidean space \mathbb{R}^3 .

In Section 5 we give some final remarks on our results; in particular we consider the case where the genus of the compact surface Σ is positive, and we show the necessity of the hypothesis that the Gauss map of S is a diffeomorphism for Theorem 4.1 and Theorem 1.3 to hold.

In the particular case that \mathcal{H} is constant, Theorem 4.1 follows from results by Hopf [Ho] when X has constant curvature, by Abresch and Rosenberg [AbR1, AbR2] when X is rotationally

symmetric, by Daniel and Mira [DM] when $X = \text{Sol}_3$, and by Meeks, Mira, Pérez and Ros [MMPR] for general X . In particular, our proof is inspired by the study in [DM, MMPR] of that constant mean curvature case.

2. Homogeneous three-manifolds

In this section we explain some basic geometric facts regarding homogeneous three-manifolds. More specific details may be consulted in [MMPR, MP].

Let \bar{M}^3 be a homogeneous, simply connected Riemannian three-manifold, and assume that \bar{M}^3 is not isometric to the product space $\mathbb{S}^2(\kappa) \times \mathbb{R}$ of a two-dimensional sphere $\mathbb{S}^2(\kappa)$ with the real line. Then \bar{M}^3 is diffeomorphic to \mathbb{R}^3 or \mathbb{S}^3 , and is isometric to a *metric Lie group*, i.e. a three-dimensional simply connected Lie group X furnished with a left invariant metric $\langle \cdot, \cdot \rangle$.

The isometry group of \bar{M}^3 has dimension six, four or three. When the dimension is six, \bar{M}^3 has constant curvature. When the dimension is four, \bar{M}^3 is rotationally symmetric, and is one of the Riemannian fibrations $\mathbb{E}^3(\kappa, \tau)$, i.e. the product spaces $\mathbb{H}^2(\kappa) \times \mathbb{R}$ and $\mathbb{S}^2(\kappa) \times \mathbb{R}$ for $\kappa \neq 0, \tau = 0$, the Heisenberg space Nil_3 for $\kappa = 0, \tau \neq 0$, and some rotational metrics on $\text{SU}(2)$ or the universal cover of $\text{SL}(2, \mathbb{R})$ if $\tau \neq 0$ and $\kappa \neq 0$. See [D1] for the details. A generic homogeneous three-manifold has isometry group of dimension three, and the identity component is generated by the group of left translations of X , when we view X as a metric Lie group.

It is important to observe that the homogeneous three-manifolds $\mathbb{R}^3, \mathbb{H}^3$ and $\mathbb{E}(\kappa, \tau)$ with $\kappa < 0$ admit more than one Lie group structure for which the metric is left invariant. That is, these homogeneous three-manifolds are isometric to at least two metric Lie groups X, X' that are non-isomorphic as Lie groups.

For computational purposes, it will be useful to divide the class of metric Lie groups X into two cases: *unimodular* metric Lie groups and *non-unimodular* metric Lie groups. (See [MP] for some equivalent definitions of unimodularity, although we will not use the concept itself, just the resulting classification of Lie groups). We note that \mathbb{R}^3 and \mathbb{S}^3 are unimodular, while \mathbb{H}^3 is non-unimodular.

2.1. Unimodular metric Lie groups. Let X be a three-dimensional unimodular metric Lie group. Then, there exists a left invariant orthonormal frame $\{E_1, E_2, E_3\}$ in X which satisfies the following structure equations:

$$(2.1) \quad [E_2, E_3] = c_1 E_1, \quad [E_3, E_1] = c_2 E_2, \quad [E_1, E_2] = c_3 E_3,$$

for certain constants $c_1, c_2, c_3 \in \mathbb{R}$, among which at most one c_i is negative. We call $\{E_1, E_2, E_3\}$ the *canonical frame* of X .

Two unimodular metric Lie groups with the same structure constants are isometric and isomorphic. Two unimodular metric Lie groups with the same signature for the triple (c_1, c_2, c_3) are isomorphic, but not isometric in general. If $c_1 = c_2 = c_3$, then X has constant curvature. If two of the constants c_i coincide, X has an isometry group of dimension four, and hence is rotationally symmetric.

The table below shows the six possible different Lie group structures depending on the signature of (c_1, c_2, c_3) . Each horizontal line corresponds to a unique Lie group structure; when all the structure constants are different, the isometry group of X is three-dimensional.

Signs of c_1, c_2, c_3	$\dim \text{Isom}(X) = 3$	$\dim \text{Isom}(X) = 4$	$\dim \text{Isom}(X) = 6$
$+, +, +$	$\text{SU}(2)$	$\mathbb{S}_{\text{Berger}}^3 = \mathbb{E}(\kappa, \tau), \kappa > 0$	$\mathbb{S}^3(\kappa)$
$+, +, -$	$\widetilde{\text{SL}}(2, \mathbb{R})$	$\mathbb{E}(\kappa, \tau), \kappa < 0$	\emptyset
$+, +, 0$	$\widetilde{\text{E}}(2)$	\emptyset	$(\widetilde{\text{E}}(2), \text{flat})$
$+, -, 0$	Sol_3	\emptyset	\emptyset
$+, 0, 0$	\emptyset	$\text{Nil}_3 = \mathbb{E}(0, \tau), \tau \neq 0$	\emptyset
$0, 0, 0$	\emptyset	\emptyset	\mathbb{R}^3

TABLE 1. All three-dimensional, simply connected unimodular metric Lie groups. Here, $\widetilde{\text{SL}}(2, \mathbb{R})$ is the universal cover of $\text{SL}(2, \mathbb{R})$, $\widetilde{\text{E}}(2)$ the universal cover of the group of orientation preserving

Euclidean isometries of \mathbb{R}^2 , Sol_3 is the universal cover of the group of orientation preserving isometries of the Lorentzian plane, Nil_3 is Heisenberg group of real upper triangular 3×3 matrices, and \mathbb{R}^3 is the abelian group.

If we write

$$(2.2) \quad \mu_1 = \frac{1}{2}(-c_1 + c_2 + c_3), \quad \mu_2 = \frac{1}{2}(c_1 - c_2 + c_3), \quad \mu_3 = \frac{1}{2}(c_1 + c_2 - c_3),$$

the Riemannian connection of (X, \langle, \rangle) is given by

$$(2.3) \quad \nabla_{E_i} E_j = \mu_i E_i \times E_j, \quad i, j \in \{1, 2, 3\}.$$

Here, \times denotes the cross product associated to \langle, \rangle and to the orientation on X defined by declaring (E_1, E_2, E_3) to be a positively oriented basis.

2.2. Non-unimodular metric Lie groups. Let X be a three-dimensional, simply connected non-unimodular metric Lie group. Then we can view X as a semi-direct product $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical metric, as we explain next.

Let A be a 2×2 matrix with trace 2, which we write in the form

$$(2.4) \quad A = A(a, b) = \begin{pmatrix} 1+a & -(1-a)b \\ (1+a)b & 1-a \end{pmatrix}, \quad a, b \in [0, \infty).$$

Then we can consider the metric Lie group $\mathbb{R}^2 \rtimes_A \mathbb{R}$ given as $(\mathbb{R}^3 \equiv \mathbb{R}^2 \times \mathbb{R}, *, \langle, \rangle)$ where:

(1) The Lie group operation $*$ is given by

$$(2.5) \quad (\mathbf{p}_1, z_1) * (\mathbf{p}_2, z_2) = (\mathbf{p}_1 + e^{z_1 A} \mathbf{p}_2, z_1 + z_2).$$

(2) The *canonical metric* \langle, \rangle is the left invariant metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$ (for the product $*$ above) defined by extending the usual inner product of \mathbb{R}^3 at the origin to the whole space through the left invariant frame $\{E_1, E_2, E_3\}$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ given by

$$(2.6) \quad E_1(x, y, z) = a_{11}(z)\partial_x + a_{21}(z)\partial_y, \quad E_2(x, y, z) = a_{12}(z)\partial_x + a_{22}(z)\partial_y, \quad E_3 = \partial_z,$$

where

$$(2.7) \quad e^{zA} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}.$$

In this way, $\{E_1, E_2, E_3\}$ becomes a left invariant orthonormal frame on $\mathbb{R}^2 \rtimes_A \mathbb{R}$, which we call the *canonical frame* of the space.

In terms of A , the Lie bracket relations are:

$$(2.8) \quad [E_1, E_2] = 0, \quad [E_3, E_1] = (1+a)E_1 + b(1+a)E_2, \quad [E_3, E_2] = b(a-1)E_1 + (1-a)E_2.$$

From there, the Levi-Civita connection of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is given by

$$(2.9) \quad \begin{array}{l} \nabla_{E_1} E_1 = (1+a)E_3 \\ \nabla_{E_2} E_1 = abE_3 \\ \nabla_{E_3} E_1 = bE_2 \end{array} \left| \begin{array}{l} \nabla_{E_1} E_2 = abE_3 \\ \nabla_{E_2} E_2 = (1-a)E_3 \\ \nabla_{E_3} E_2 = -bE_1 \end{array} \right| \begin{array}{l} \nabla_{E_1} E_3 = -(1+a)E_1 - abE_2 \\ \nabla_{E_2} E_3 = -abE_1 - (1-a)E_2 \\ \nabla_{E_3} E_3 = 0. \end{array}$$

The *Cheeger constant* $\text{Ch}(\mathbb{R}^2 \rtimes_A \mathbb{R})$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is $\text{trace}(A) = 2$. We also note that every leaf of the foliation $\mathcal{F} = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ has constant mean curvature $H = \text{trace}(A)/2 = 1$ with respect to the unit normal vector field E_3 . In particular, by the mean curvature comparison principle, there are no immersed compact surfaces in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with mean curvature function $|H| > 1$ at every point. In other words, the *critical mean curvature* of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ is 1 (see [MP, MMPR, MMPR2]).

This construction of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ that we have just carried out recovers (up to homothety) all non-unimodular metric Lie groups:

Fact (see [MP]): Let X be a simply connected non-unimodular three-dimensional metric Lie group, with its metric rescaled so that $\text{Ch}(X) = 2$. Then X is isomorphic and isometric to the semi-direct $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric and A given by (2.4) for some $a, b \in [0, \infty)$,

The hyperbolic three-space \mathbb{H}^3 of constant curvature -1 is the semi-direct product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $A = I_2$. Similarly, if $a = 1, b = 0$ we recover the product space $\mathbb{H}^2(-4) \times \mathbb{R}$.

2.3. The Gauss map for surfaces in metric Lie groups. Let $\psi : \Sigma \rightarrow X$ be an immersed oriented surface in a metric Lie group X , and let $N : \Sigma \rightarrow TX$ denote its unit normal. Note that for any $x \in X$ the left translation $l_x : X \rightarrow X$ is an isometry of X . Thus, for every $p \in \Sigma$ there exists a unique unit vector $\nu(p) \in T_e X$, $|\nu(p)| = 1$, such that

$$dl_{\psi(p)}(\nu(p)) = N(p), \quad \forall p \in \Sigma,$$

where e denotes the identity element of X .

DEFINITION 2.1. We call the map $\nu : \Sigma \rightarrow \mathbb{S}^2 = \{v \in T_e X : |v| = 1\}$ the Gauss map of the oriented surface $\psi : \Sigma \rightarrow X$.

The Gauss map ν can be easily written in coordinates as follows: let $\{E_1, E_2, E_3\}$ be a left invariant orthonormal frame of X , and write $N = \sum_{i=1}^3 \nu_i E_i$ for the unit normal N of ψ . Then the Gauss map $\nu : \Sigma \rightarrow \mathbb{S}^2 \subset T_e X$ is written with respect to the orthonormal basis of the Lie algebra $\{(E_1)_e, (E_2)_e, (E_3)_e\}$ as $\nu = (\nu_1, \nu_2, \nu_3) : \Sigma \rightarrow \mathbb{S}^2$.

Note that if X is the Euclidean three-space \mathbb{R}^3 , we have $dl_x = \text{Id}$ for every $x \in \mathbb{R}^3$; thus, the Gauss map ν in Definition 2.1 is the natural extension to metric Lie groups of the usual Gauss map of surfaces in \mathbb{R}^3 .

Also note that we can extend the notion of surfaces with prescribed mean curvature in terms of the Gauss map in \mathbb{R}^3 (see Definition 1.2) to the case of metric Lie groups:

DEFINITION 2.2. Let X be a metric Lie group, and $\mathcal{H} \in C^1(\mathbb{S}^2)$. An immersed oriented surface Σ in X with Gauss map $\nu : \Sigma \rightarrow \mathbb{S}^2$ is said to have *prescribed mean curvature* \mathcal{H} if

$$H_\Sigma(p) = \mathcal{H}(\nu(p))$$

for every $p \in \Sigma$, where H_Σ is the mean curvature of Σ .

REMARK 2.3. When X is the hyperbolic three-space \mathbb{H}^3 or the sphere \mathbb{S}^3 , the left invariant Gauss map $\nu : \Sigma \rightarrow \mathbb{S}^2$ of Definition 2.2 is usually called the normal Gauss map. The problem of prescribing a mean curvature function and the normal Gauss map in \mathbb{H}^3 or \mathbb{S}^3 has been treated, for instance, in [Ko, AA1, AA2].

A different but also natural choice of Gauss map for surfaces in \mathbb{H}^3 (which we do not treat here) is the hyperbolic Gauss map; see [EGM] for the study of a Christoffel-Minkowski problem in \mathbb{H}^{n+1} in terms of the hyperbolic Gauss map.

REMARK 2.4. In \mathbb{R}^3 the Gauss map of a surface $S \subset \mathbb{R}^3$ is a diffeomorphism into \mathbb{S}^2 if and only if S is a strictly convex ovaloid. In particular, S is embedded. This is not true in general when we substitute \mathbb{R}^3 by a metric Lie group X . For instance, in some homogeneous three-manifolds diffeomorphic (but not isometric) to \mathbb{S}^3 there exist constant mean curvature spheres which are not embedded, but whose Gauss maps are diffeomorphisms into \mathbb{S}^2 (see [MMPR, To]).

2.4. The potential function of X . The next definition is a slight reformulation of the concept of H -potential of a three-dimensional metric Lie group in [MMPR]. It will play an important role in our computations in the next two sections.

DEFINITION 2.5. The potential function of the space X is the map $R(H, q) : \mathbb{R} \times \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ given by

$$(2.10) \quad R(H, q) = H(1 + |q|^2)^2 + \Theta(q),$$

where:

(1) If X is unimodular, then

$$\Theta(q) = -\frac{i}{2} (\mu_2 |1 + q^2|^2 + \mu_1 |1 - q^2|^2 + 4\mu_3 |q|^2),$$

where $\mu_1, \mu_2, \mu_3 \in \mathbb{R}$ are the related numbers defined in (2.2).

- (2) If X is non-unimodular, and we rescale its metric as explained in Subsection 2.2 so that $\text{Ch}(X) = 2$, then

$$\Theta(q) = -(1 - |q|^4) - a(q^2 - \bar{q}^2) - ib(2|q|^2 - a(q^2 + \bar{q}^2))$$

where $a, b \geq 0$ are the constants appearing in (2.4) when we view X as the semi-direct product $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

We will say that the potential R for X has a zero at $q_0 = \infty \in \bar{\mathbb{C}}$ if $\lim_{q \rightarrow \infty} R(H, q)/|q|^4 = 0$ for every H .

The zeros of R are related to the existence of two-dimensional subgroups of X , as follows: $R(H_0, q_0) = 0$ in X if and only if there exists a two dimensional subgroup in X with (constant) Gauss map q_0 and (constant) mean curvature H_0 (see Corollary 3.17 in [MP]).

The potential function R has no zeros if X is compact (i.e. if X is diffeomorphic to \mathbb{S}^3). If X is unimodular and $H_0 \neq 0$, then $R(H_0, q) \neq 0$ for all $q \in \bar{\mathbb{C}}$. If X is non-unimodular, rescaled so that $\text{Ch}(X) = 2$ as explained in Subsection 2.2, and if $|H_0| > 1$, then $R(H_0, q) \neq 0$ for every $q \in \bar{\mathbb{C}}$. Thus we have:

LEMMA 2.6. Assume that X is not compact, and let $\mathfrak{h}_0(X) \geq 0$ be the number given by

$$\begin{cases} \mathfrak{h}_0(X) = 0 & \text{if } X \text{ is unimodular.} \\ \mathfrak{h}_0(X) = 1 & \text{if } X \text{ is non-unimodular, rescaled to } \text{Ch}(X) = 2. \end{cases}$$

Then $R(H, q) \neq 0$ for every $q \in \bar{\mathbb{C}}$ and every H with $|H| > \mathfrak{h}_0(X)$.

3. An elliptic PDE for the Gauss map

A theorem by Kenmotsu [Ke] proves that a necessary and sufficient condition for a map $g : \Sigma \rightarrow \bar{\mathbb{C}}$ from a simply connected Riemann surface Σ to be the Gauss map of a conformal immersion $\psi : \Sigma \rightarrow \mathbb{R}^3$ with a given mean curvature function $H : \Sigma \rightarrow (0, \infty)$ is that

$$g_{z\bar{z}} = \frac{2\bar{g}}{1 + |g|^2} g_z g_{\bar{z}} + \frac{H_{\bar{z}}}{H} g_z,$$

for any conformal parameter z of Σ . Moreover, if this equation holds, the immersion ψ can be recovered from g, H by an integral representation formula.

In this section we extend this theorem to the general case where the ambient space is an arbitrary simply connected homogeneous three-manifold X not isometric to $\mathbb{S}^2(\kappa) \times \mathbb{R}$. This also extends the Weierstrass type representation for CMC surfaces in homogeneous manifolds in [MMPR], and the work by Aiyama and Akutagawa [AA1, AA2] for prescribed non-constant mean curvature surfaces in \mathbb{H}^3 and \mathbb{S}^3 .

In the next result Σ denotes a Riemann surface and z an arbitrary conformal parameter on Σ . We identify the Gauss map $\nu : \Sigma \rightarrow \mathbb{S}^2$ of an oriented surface $\psi : \Sigma \rightarrow X$ with its south pole stereographic projection g , i.e. if $N = \sum \nu_i E_i$ with respect to the canonical frame $\{E_1, E_2, E_3\}$ of X , then

$$(3.1) \quad g = \frac{\nu_1 + i\nu_2}{1 + \nu_3} : \Sigma \rightarrow \bar{\mathbb{C}}.$$

Also, $R(H, q) : \mathbb{R} \times \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ will denote the potential function of X (see Definition 2.5).

THEOREM 3.1. Let $\psi : \Sigma \rightarrow X$ be a conformally immersed oriented surface, and let $H : \Sigma \rightarrow \mathbb{R}$ and $g : \Sigma \rightarrow \bar{\mathbb{C}}$ denote its mean curvature and Gauss map, respectively. Then (g, H) satisfy the conformally invariant complex elliptic PDE

$$(3.2) \quad g_{z\bar{z}} = \frac{R_q}{R}(H, g) g_z g_{\bar{z}} + \left(\frac{R_{\bar{q}}}{R} - \frac{\overline{R_q}}{R} \right) (H, g) |g_z|^2 + \frac{R_H}{R}(H, g) H_{\bar{z}} g_z,$$

at all points $p \in \Sigma$ with $R(H(p), g(p)) \neq 0$.

Conversely, let Σ be simply connected, and let $H : \Sigma \rightarrow \mathbb{R}$ and $g : \Sigma \rightarrow \bar{\mathbb{C}}$ satisfy:

- (1) $R(H(p), g(p)) \neq 0$ for every $p \in \Sigma$.
- (2) $g_z(p) \neq 0$ for every¹ $p \in \Sigma$.

¹If $g(p) = \infty$, the condition $g_z(p) \neq 0$ should be interpreted as $\lim_{z \rightarrow p} g_z(p)/g(p)^2 \neq 0$.

(3) (H, g) are a solution to (3.2).

Then there exists a conformal immersion $\psi : \Sigma \rightarrow X$, unique up to left translations, with Gauss map g and mean curvature H .

PROOF. We will only prove the result for the case that X is unimodular; the argument when X is non-unimodular is exactly the same but some intermediate expressions are different because of the difference between the potential functions of unimodular and non-unimodular spaces.

Let $\psi : \Sigma \rightarrow X$ be a conformal immersion with unit normal $N : \Sigma \rightarrow TX$, and denote $\langle d\psi, d\psi \rangle = \lambda |dz|^2$, where z is a local conformal parameter on Σ . Let $\{E_1, E_2, E_3\}$ be the canonical left invariant orthonormal frame of X , as explained in Section 2, $\nu : \Sigma \rightarrow \mathbb{S}^2$ be the Gauss map of ψ and g be the Gauss map after stereographic projection (3.1). We will work around a point $p \in \Sigma$ where $g \neq 0, \infty$ and $R(H, g) \neq 0$. We can write

$$(3.3) \quad \psi_z = \sum_{i=1}^3 A_i (E_i \circ \psi), \quad \psi_{\bar{z}} = \sum_{i=1}^3 \overline{A_i} (E_i \circ \psi), \quad N = \sum_{i=1}^3 \nu_i (E_i \circ \psi),$$

for smooth functions $A_i : \Sigma \rightarrow \mathbb{C}$, $\nu_i : \Sigma \rightarrow \mathbb{R}$, $i = 1, 2, 3$. Noting that, from (3.1),

$$(3.4) \quad (\nu_1, \nu_2, \nu_3) = \frac{1}{1 + |g|^2} (g + \bar{g}, i(\bar{g} - g), 1 - |g|^2),$$

it follows easily from the metric relations $\langle \psi_z, \nu \rangle = \langle \psi_{\bar{z}}, \nu \rangle = 0$, $\langle \nu, \nu \rangle = 1$ that, if we denote $A_3 = \eta/2$, then

$$(3.5) \quad A_1 = \frac{\eta}{4} \left(\bar{g} - \frac{1}{\bar{g}} \right), \quad A_2 = \frac{i\eta}{4} \left(\bar{g} + \frac{1}{\bar{g}} \right), \quad A_3 = \frac{\eta}{2}.$$

From here,

$$(3.6) \quad \lambda = 2 \sum_{i=1}^3 |A_i|^2 = \frac{(1 + |g|^2)^2 |\eta|^2}{4|g|^2}.$$

If we let ∇ denote the Levi-Civita connection of X , a classical computation from the Gauss-Weingarten equations gives

$$(3.7) \quad \nabla_{\psi_{\bar{z}}} \psi_z = \frac{\lambda H}{2} \nu.$$

If we now express ψ_z and $\psi_{\bar{z}}$ as in (3.3) and use the relations between ∇ and $\{E_1, E_2, E_3\}$ given in (2.3), we can write (3.7) in coordinates with respect to $\{E_1, E_2, E_3\}$. Let us use brackets to denote coordinates in the $\{E_1, E_2, E_3\}$ basis. Then, we obtain

$$(3.8) \quad \begin{aligned} [(A_1)_{\bar{z}}, (A_2)_{\bar{z}}, (A_3)_{\bar{z}}] &= -\sum_{i,j} \overline{A_i} A_j \nabla_{E_i} E_j + \frac{\lambda H}{2} [\nu_1, \nu_2, \nu_3] \\ &= [\mu_3 A_2 \overline{A_3} - \mu_2 A_3 \overline{A_2}, \mu_3 A_1 \overline{A_3} - \mu_1 A_3 \overline{A_1}, \mu_2 A_1 \overline{A_2} - \mu_1 A_2 \overline{A_1}] \\ &\quad + \frac{\lambda H}{2} [\nu_1, \nu_2, \nu_3]. \end{aligned}$$

The third equation in (3.8) gives, using (3.5) and (3.4),

$$(3.9) \quad \frac{4\eta_{\bar{z}}}{|\eta|^2} = \frac{H(1 - |g|^4)}{|g|^2} - \frac{i}{2} \left\{ \mu_1 \left(\frac{1}{\bar{g}} + \bar{g} \right) \left(-\frac{1}{g} + g \right) - \mu_2 \left(\frac{1}{\bar{g}} - \bar{g} \right) \left(\frac{1}{g} + g \right) \right\}$$

A similar process can be done with (1st) + i (2nd) equation in (3.8), using again (3.5) and (3.4). In this way we obtain

$$(3.10) \quad \frac{4\eta_{\bar{z}}}{|\eta|^2} = \frac{4\overline{g_z}}{\bar{g}\eta} - 2H(1 + |g|^2) - i \left\{ \mu_1 \left(|g|^2 - \frac{\bar{g}}{g} \right) + \mu_2 \left(|g|^2 + \frac{\bar{g}}{g} \right) + 2\mu_3 \right\}.$$

By comparing (3.10) with (3.9), a computation provides the following expression for η :

$$(3.11) \quad \eta = \frac{4\overline{g_z}}{R(H, g)}.$$

Differentiating (3.11), we get

$$(3.12) \quad \frac{\eta_{\bar{z}}}{\eta} = \frac{\bar{g}_z}{\bar{g}} + \frac{g_{z\bar{z}}}{g_z} - \frac{R_q}{R}(H, g)g_{\bar{z}} - \frac{R_{\bar{q}}}{R}(H, g)\bar{g}_z - \frac{R_H}{R}(H, g)H_{\bar{z}}.$$

Finally, comparing (3.12) with (3.9) and using (the conjugate of) (3.11), we obtain the Gauss map equation (3.2). By smoothness, it also holds when $g = 0, \infty$ (that (3.2) extends to the points where $g = \infty$ can be easily checked by working at those points with the smooth function $\varphi = 1/g$). This completes the proof of the first statement of the theorem.

The proof of the converse statement follows by a computation using the Frobenius theorem. Specifically, the argument follows very closely the proof of Theorem 3.7 in [MMPR], which covers the case where H is constant. We give an outline next.

Let $g : \Sigma \rightarrow \mathbb{C}$ and $H : \Sigma \rightarrow \mathbb{R}$ satisfy conditions (1), (2), (3) as in the statement of Theorem 3.1, and assume that Σ is simply connected. We define $A_1, A_2, A_3 : \Sigma \rightarrow \mathbb{C}$ as in (3.5), where η is given by (3.11); that all the A_i 's have finite value at every $p \in \Sigma$ follows from the fact that $R(H, g)/|g|^4$ has a finite limit at $g = \infty$. We compute

$$(3.13) \quad \begin{aligned} (A_1)_{\bar{z}} &= \frac{\bar{g}^2 - 1}{R(H, g)}g_{z\bar{z}} - \frac{\bar{g}^2 - 1}{R(H, g)^2}(R(H, g))_{\bar{z}}g_z + \frac{2\bar{g}}{R(H, g)}|g_z|^2 \\ &= \frac{\bar{g}^2 - 1}{R(H, g)}g_{z\bar{z}} - \frac{\bar{g}^2 - 1}{R(H, g)^2}(R_q(H, g)g_{\bar{z}} + R_{\bar{q}}(H, g)\bar{g}_z + R_H(H, g)H_{\bar{z}})g_z \\ &\quad + \frac{2\bar{g}}{R(H, g)}|g_z|^2. \end{aligned}$$

Using in this expression that g is a solution to (3.2), we get

$$(3.14) \quad (A_1)_{\bar{z}} = \frac{|g_z|^2}{|R(H, g)|^2} \left(\overline{2gR(H, g) - R_q(H, g)(g^2 - 1)} \right).$$

Observe that the derivative $H_{\bar{z}}$ in (3.13) cancels in (3.14) with the $H_{\bar{z}}$ appearing after substituting $g_{z\bar{z}}$ by its value in (3.2). The same happens if we work with A_2 or A_3 instead of A_1 .

Note that (3.14) is exactly the same formula as (3.5) in [MMPR] (with the change of notation of substituting $R(g)$ there by $R(H, g)$ here). In other words, formula (3.5) in [MMPR] also holds when H is not constant.

At this point, the rest of the proof of the direct statement in Theorem 3.7 of [MMPR] never uses again that H is constant. Thus, it translates essentially word by word to our situation, up to the change of notation $R(g) \leftrightarrow R(H, g)$ explained above. This finishes the proof of Theorem 3.1. \square

3.1. Remarks.

- (1) In the proof of Theorem 3.1 we established that the metric of a conformal immersion $\psi : \Sigma \rightarrow X$ with mean curvature $H : \Sigma \rightarrow \mathbb{R}$ and Gauss map g is

$$(3.15) \quad \langle d\psi, d\psi \rangle = \lambda |dz|^2, \quad \lambda = \frac{4(1 + |g|^2)^2}{|R(H, g)|^2} |g_z|^2.$$

Thus, if $R(H(p), g(p)) \neq 0$ for some $p \in \Sigma$, then $g_z(p) \neq 0$.

- (2) In particular, it follows from Lemma 2.6 that if X is compact, or if the mean curvature of $\psi : \Sigma \rightarrow X$ satisfies $|H| > \mathfrak{h}_0(X)$ when X is non-compact, then $g_z \neq 0$ everywhere on Σ .
- (3) The converse of Theorem 3.1 provides a Weierstrass representation formula which recovers a conformal immersion $\psi : \Sigma \rightarrow X$ (with $|H| > \mathfrak{h}_0(X)$ if X is non-compact) from its mean curvature function H and its Gauss map g . Specifically, if Σ is simply connected and $(g, H) : \Sigma \rightarrow \mathbb{C} \times \mathbb{R}$ satisfy conditions (1), (2), (3) in the statement of Theorem 3.1, then we can define $A_i : \Sigma \rightarrow \mathbb{C}$, $i = 1, 2, 3$, by

$$A_1 = \frac{\eta}{4} \left(\bar{g} - \frac{1}{\bar{g}} \right), \quad A_2 = \frac{i\eta}{4} \left(\bar{g} + \frac{1}{\bar{g}} \right), \quad A_3 = \frac{\eta}{2}, \quad \eta = \frac{4\bar{g}g_z}{R(g)}.$$

and the immersion $\psi : \Sigma \rightarrow X$ with mean curvature H and Gauss map g may be recovered from (g, H) by integrating

$$\psi_z = \sum_{i=1}^3 A_i(E_i \circ \psi)$$

on Σ . Here, $\{E_1, E_2, E_3\}$ is the canonical frame of X . The final formula for ψ involves intricate integral expressions in terms of the coordinates of ψ and the structure constants of X , and so will be omitted here.

- (4) As a consequence of the previous remark we have the following *uniqueness result*: Let $\psi_1, \psi_2 : \Sigma \rightarrow X$ be two conformal immersions with the same mean curvature $H : \Sigma \rightarrow \mathbb{R}$, the same Gauss map $g : \Sigma \rightarrow \mathbb{C}$, and so that $R(H(z), g(z)) \neq 0$ for every $z \in \Sigma$. Then $\psi_2 = L \circ \psi_1$ for some left translation $L : X \rightarrow X$.
- (5) The usual Hopf differential $P dz^2$ of $\psi : \Sigma \rightarrow X$ is defined as

$$P = -\langle \nabla_{\psi_z} N, \psi_z \rangle,$$

and can be computed using (3.4), (3.5), (3.11) as follows:

$$\begin{aligned} P &= -\sum_{i=1}^3 A_i(\nu_i)_z - \sum_{i,j,k=1}^3 A_i \nu_j A_k \langle \nabla_{E_i} E_j, E_k \rangle \\ (3.16) \quad &= \frac{2}{R(H, g)} g_z \bar{g}_z - \sum_{i,j,k=1}^3 \gamma_{ij}^k A_i \nu_j A_k, \end{aligned}$$

where $\gamma_{ij}^k := \langle \nabla_{E_i} E_j, E_k \rangle$ is a constant depending on the Lie algebra structure of X . Thus, using again the relations (3.4), (3.5), (3.11) we see that P can be written in the form

$$(3.17) \quad P = \frac{2}{R(H, g)} g_z \bar{g}_z + U(g, \bar{g}) g_z^2,$$

where $U(g, \bar{g})$ is a rational expression in g, \bar{g} whose coefficients depend on γ_{ij}^k .

4. Prescribed mean curvature spheres: proof of Theorem 4.1

Let X be a three-dimensional metric Lie group, and $\mathcal{H} \in C^1(\mathbb{S}^2)$. In this section we prove:

THEOREM 4.1. *Let S be an immersed sphere in X with prescribed mean curvature $\mathcal{H} \in C^1(\mathbb{S}^2)$, $\mathcal{H} > 0$, and assume that the Gauss map $\nu : S \rightarrow \mathbb{S}^2$ is a diffeomorphism.*

Then any other immersed sphere Σ in X with prescribed mean curvature \mathcal{H} is a left translation of S . In particular, Σ and S are congruent in X .

REMARK 4.2. *Theorem 4.1 clearly implies Theorem 1.3. Note that under the hypotheses of Theorem 1.3, the prescribed function $\mathcal{H} \in C^1(\mathbb{S}^2)$ needs to be positive at every point.*

To begin the proof of Theorem 4.1, assume that there exists an immersed sphere S in X with prescribed mean curvature \mathcal{H} , and whose Gauss map $\nu : S \rightarrow \mathbb{S}^2$ is a diffeomorphism. Let $G = \pi \circ \nu : S \rightarrow \mathbb{C}$ where π denotes the stereographic projection from the south pole, i.e. if the unit normal N of ψ is expressed as $N = \sum_i \nu_i E_i$ with respect to the canonical frame $\{E_1, E_2, E_3\}$ of X , then

$$G = \frac{\nu_1 + i\nu_2}{1 + \nu_3} : S \rightarrow \mathbb{C}.$$

Then G is an orientation preserving diffeomorphism, and so $|G_z|^2 - |G_{\bar{z}}|^2 > 0$. In particular $G_z \neq 0$ at every point. By (3.15), the potential function $R(H, q)$ of X does not vanish at the points of the form $(H_S(p), G(p))$, where $p \in S$ and H_S is the mean curvature function of S . So, as G is a diffeomorphism, $R(\mathcal{H}(q), q) \neq 0$ for every $q \in \mathbb{C}$.

Theorem 4.1 follows easily from the following result of independent interest:

THEOREM 4.3. *In the conditions of Theorem 4.1, there exists a complex quadratic differential $Q_{\mathcal{H}} dz^2$ defined for any immersed surface $\psi : \Sigma \rightarrow X$ with prescribed mean curvature \mathcal{H} , so that:*

- (1) $Q_{\mathcal{H}} dz^2$ vanishes identically on S .

- (2) If $Q_{\mathcal{H}} dz^2$ vanishes identically on $\psi : \Sigma \rightarrow X$, then $\psi(\Sigma)$ is a left translation in X of an open subset of S .
- (3) If $Q_{\mathcal{H}} dz^2$ does not vanish identically on $\psi : \Sigma \rightarrow X$, then the zeros of $Q_{\mathcal{H}} dz^2$ are all isolated and of negative index on Σ .

Indeed, by the Poincaré-Hopf theorem, a complex quadratic differential on $\bar{\mathbb{C}}$ cannot have only isolated zeros of negative index. Thus Theorem 4.1 follows from Theorem 4.3.

PROOF OF THEOREM 4.3. Define

$$(4.1) \quad \mathcal{M}(q) = \frac{1}{R(\mathcal{H}(q), q)} : \bar{\mathbb{C}} \rightarrow \mathbb{C},$$

which by the discussion above takes finite values, and let $\mathcal{L}(q) : \bar{\mathbb{C}} \rightarrow \mathbb{C}$ be

$$(4.2) \quad \mathcal{L}(q) = -\frac{\bar{G}_z}{G_z}(G^{-1}(q))\mathcal{M}(q).$$

We define for any conformally immersed surface $\psi : \Sigma \rightarrow X$ with prescribed mean curvature \mathcal{H} the complex quadratic differential $Q_{\mathcal{H}} dz^2$ on Σ given by

$$(4.3) \quad Q_{\mathcal{H}} = \mathcal{L}(g)g_z^2 + \mathcal{M}(g)g_z\bar{g}_z.$$

Here z is an arbitrary conformal parameter of Σ , and $g : \Sigma \rightarrow \bar{\mathbb{C}}$ is the Gauss map of ψ .

We divide the proof of Theorem 4.3 into several claims.

Claim 1: $Q_{\mathcal{H}} dz^2$ is a well defined complex quadratic differential on any surface $\psi : \Sigma \rightarrow X$ with prescribed mean curvature \mathcal{H} .

PROOF OF CLAIM 1. The invariance of $Q_{\mathcal{H}} dz^2$ under conformal changes of coordinates is clear. Besides, it follows from the discussion above that $\mathcal{L}(g)$ and $\mathcal{M}(g)$ take finite values at points $p \in \Sigma$ where $g(p) \neq \infty$. So, we only need to check that $Q_{\mathcal{H}}$ can be defined even when $g(p) = \infty$. To do so first observe that, since the potential function $R(H, q)$ satisfies that $R(H, q)/|q|^4$ has a smooth extension to $q = \infty$ for every $H \in \mathbb{R}$, then $|q|^4\mathcal{M}(q)$ also has a smooth extension to $q = \infty$. Also, from the definition of \mathcal{L} in (4.2) and the fact that $G_z \neq 0$ (see the footnote in the statement of Theorem 3.1 for the meaning of this condition at points where $G = \infty$), we can deduce that $|q|^4\mathcal{L}(q)$ also has a finite limit as $q \rightarrow \infty$. From here, one can easily show that $Q_{\mathcal{H}} dz^2$ is well defined even at points $p \in \Sigma$ where the Gauss map g of ψ satisfies $g(p) = \infty$. \square

Claim 2: $Q_{\mathcal{H}} dz^2$ vanishes identically on S . And conversely, if $Q_{\mathcal{H}} dz^2$ vanishes identically for some surface $\psi : \Sigma \rightarrow X$ of prescribed mean curvature \mathcal{H} , then $\psi(\Sigma)$ is a left translation of an open subset of S .

PROOF OF CLAIM 2. The first assertion is trivial by the very definition of $Q_{\mathcal{H}}$. The converse statement can be proved using the idea in [DM, Lemma 4.6]. We include a proof here for the sake of completeness.

Let $\psi : \Sigma \rightarrow X$ be a conformal immersion with $Q_{\mathcal{H}} \equiv 0$, and let $g : \Sigma \rightarrow \bar{\mathbb{C}}$ be its Gauss map. Note that $g_z \neq 0$ at every point. Also, as $G : S \equiv \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ is an orientation preserving diffeomorphism, we have $|G_z|^2 - |G_{\bar{z}}|^2 > 0$. Define $\phi : G^{-1} \circ g : \Sigma \rightarrow S \equiv \bar{\mathbb{C}}$. An elementary computation shows that

$$(4.4) \quad \phi_{\bar{z}} = \frac{1}{|G_z|^2 - |G_{\bar{z}}|^2} (\overline{G_z}g_z - G_{\bar{z}}\bar{g}_z),$$

where $G_z, G_{\bar{z}}$ are evaluated at $\phi(z)$ for every $z \in \Sigma$. Since by hypothesis

$$\mathcal{L}(g)g_z + \mathcal{M}(g)\bar{g}_z = 0,$$

we can rewrite (4.4) as

$$(4.5) \quad \phi_{\bar{z}} = \frac{1}{|G_z|^2 - |G_{\bar{z}}|^2} \left(-\frac{\bar{\mathcal{L}}(g)}{\bar{\mathcal{M}}(g)} \overline{G_z} - G_{\bar{z}} \right) \bar{g}_z,$$

where again $G_z, G_{\bar{z}}$ are evaluated at $\phi(z)$. Observe now that $g = G \circ \phi$ and that $\mathcal{L}(G)G_z + \mathcal{M}(G)\overline{G_z} = 0$ since $Q_{\mathcal{H}} \equiv 0$ on S . This implies by (4.5) that $\phi_{\bar{z}} = 0$, i.e. ϕ is holomorphic.

Therefore, up to a local conformal change of coordinates in Σ we can assume that $G = g$ on a neighborhood $U \subset \bar{\mathbb{C}}$ of an arbitrary point z_0 of Σ . In particular the mean curvatures of ψ and S coincide on U since ψ and S have the same prescribed mean curvature function \mathcal{H} and the same Gauss map. So, by the uniqueness in Theorem 3.1 (see Remark 4 in Subsection 3.1), we have that $\psi(U)$ differs from an open set of S by a left translation in X . A simple continuation argument shows that the same is true for $\psi(\Sigma)$. \square

Claim 3: The maps \mathcal{L}, \mathcal{M} defined in (4.2), (4.1) satisfy the following PDEs on $\bar{\mathbb{C}}$:

$$(4.6) \quad \mathcal{M}_{\bar{q}} + (\overline{\mathcal{A}} + \mathcal{B})\mathcal{M} = \mathcal{H}_{\bar{q}}(1 + |q|^2)^2 |\mathcal{M}|^2.$$

$$(4.7) \quad (\mathcal{L}_q + 2\mathcal{A}\mathcal{L})\overline{\mathcal{L}} = (\mathcal{L}_{\bar{q}} + 2\mathcal{B}\mathcal{L} + \overline{\mathcal{B}}\mathcal{M} - \mathcal{L}\mathcal{H}_{\bar{q}}\overline{\mathcal{M}}(1 + |q|^2)^2)\overline{\mathcal{M}},$$

Here $\mathcal{A}, \mathcal{B} : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ are defined as

$$(4.8) \quad \mathcal{A}(q) = -\frac{\mathcal{M}_q}{\mathcal{M}}(q), \quad \mathcal{B}(q) = \left(\frac{\overline{\mathcal{M}_q}}{\overline{\mathcal{M}}} - \frac{\mathcal{M}_{\bar{q}}}{\mathcal{M}} \right)(q) + (1 + |q|^2)^2 \mathcal{H}_{\bar{q}}(q) \overline{\mathcal{M}(q)}.$$

PROOF OF CLAIM 3. To start, let us consider an arbitrary conformally immersed surface $\psi : \Sigma \rightarrow X$ with prescribed mean curvature \mathcal{H} , and let $g : \Sigma \rightarrow \bar{\mathbb{C}}$ denote its Gauss map. As $R(\mathcal{H}(q), q) \neq 0$ for every $q \in \bar{\mathbb{C}}$, we have by (3.15) that $g_z \neq 0$ on Σ . Let H be the mean curvature of ψ , given by $H = \mathcal{H} \circ g$. Differentiating,

$$H_{\bar{z}} = \mathcal{H}_q(g)g_{\bar{z}} + \mathcal{H}_{\bar{q}}(g)\overline{g_z}.$$

From here, a computation shows that the Gauss map equation (3.2) for g can be written in this situation as

$$(4.9) \quad g_{z\bar{z}} = \mathcal{A}(g)g_z g_{\bar{z}} + \mathcal{B}(g)|g_z|^2,$$

where $\mathcal{A}(q), \mathcal{B}(q) : \bar{\mathbb{C}} \rightarrow \bar{\mathbb{C}}$ are given by (4.8).

A direct computation shows that \mathcal{M} satisfies (4.6). In order to check that \mathcal{L} satisfies (4.7) we first observe that

$$(4.10) \quad \mathcal{L}(G)G_z + \mathcal{M}(G)\overline{G_z} = 0$$

on $S \equiv \bar{\mathbb{C}}$, since $Q_{\mathcal{H}}$ vanishes identically on S . Differentiating (4.10) with respect to \bar{z} and using (4.6) together with the fact that G satisfies (4.9) (since S has prescribed mean curvature \mathcal{H}), we obtain

$$\begin{aligned} 0 &= (\mathcal{L}_q(G)G_{\bar{z}} + \mathcal{L}_{\bar{q}}(G)\overline{G_z})G_z + \mathcal{L}(G)G_{z\bar{z}} \\ &\quad + (\mathcal{M}_q(G)G_{\bar{z}} + \mathcal{M}_{\bar{q}}(G)\overline{G_z})\overline{G_z} + \mathcal{M}(G)\overline{G_{z\bar{z}}} \\ &= \{\mathcal{L}_q + \mathcal{A}\mathcal{L}\}G_z G_{\bar{z}} + \{\mathcal{L}_{\bar{q}} + \mathcal{B}\mathcal{L} + \overline{\mathcal{B}}\mathcal{M}\}|G_z|^2 \\ &\quad - \{\mathcal{A}\mathcal{M}\}|G_{\bar{z}}|^2 + \{-\mathcal{B}\mathcal{M} + \mathcal{H}_{\bar{q}}|\mathcal{M}|^2(1 + |q|^2)^2\}\overline{G_z G_{\bar{z}}}, \end{aligned}$$

where the functions between brackets are all evaluated at $q = G(z)$. Using now the relation (4.10) in this equation, we arrive at

$$\{\mathcal{L}_q + 2\mathcal{A}\mathcal{L}\}G_z G_{\bar{z}} + \{\mathcal{L}_{\bar{q}} + 2\mathcal{B}\mathcal{L} + \overline{\mathcal{B}}\mathcal{M} - \mathcal{L}\mathcal{H}_{\bar{q}}\overline{\mathcal{M}}(1 + |q|^2)^2\}G_z \overline{G_z} = 0.$$

If we use now the conjugate of (4.10) and the fact that $G_z \neq 0$, the previous equation can be reduced to

$$(4.11) \quad (\mathcal{L}_q + 2\mathcal{A}\mathcal{L})\overline{\mathcal{L}} = (\mathcal{L}_{\bar{q}} + 2\mathcal{B}\mathcal{L} + \overline{\mathcal{B}}\mathcal{M} - \mathcal{L}\mathcal{H}_{\bar{q}}\overline{\mathcal{M}}(1 + |q|^2)^2)\overline{\mathcal{M}},$$

with all functions evaluated at $q = G(z)$. Since G is a diffeomorphism, we deduce that (4.7) holds. \square

Claim 4: If $Q_{\mathcal{H}} dz^2$ does not vanish identically on a surface $\psi : \Sigma \rightarrow X$ with prescribed mean curvature \mathcal{H} , then it only has isolated zeros of negative index.

PROOF OF CLAIM 4. Let $\psi : \Sigma \rightarrow X$ be a conformal immersion with prescribed mean curvature \mathcal{H} , and let $g : \Sigma \rightarrow \bar{\mathbb{C}}$ denote its Gauss map. Recall that g satisfies the PDE (4.9), and that $R(\mathcal{H}(q), q) \neq 0$ for every $q \in \bar{\mathbb{C}}$. So, in particular, $g_z \neq 0$ at all points of Σ .

Let $Q_{\mathcal{H}} dz^2$ denote the complex quadratic differential defined in (4.3). By differentiating $Q_{\mathcal{H}}$ with respect to \bar{z} and using (4.9), we arrive at

$$(4.12) \quad \begin{aligned} (Q_{\mathcal{H}})_{\bar{z}} &= \{\mathcal{L}_q + 2\mathcal{A}\mathcal{L}\}g_{\bar{z}}g_z^2 + \{\mathcal{L}_{\bar{q}} + 2\mathcal{B}\mathcal{L} + \bar{\mathcal{B}}\mathcal{M}\}g_z|g_z|^2 \\ &\quad + \{\mathcal{M}_q + \mathcal{A}\mathcal{M}\}g_z|g_{\bar{z}}|^2 + \{\mathcal{M}_{\bar{q}} + (\bar{\mathcal{A}} + B)\mathcal{M}\}\bar{g}_z|g_z|^2, \end{aligned}$$

where the quantities in brackets are evaluated at $g(z)$ for every $z \in \Sigma$. As $\mathcal{M}_q + \mathcal{A}\mathcal{M} = 0$ by definition of \mathcal{A} , and \mathcal{M} satisfies (4.6), we obtain from (4.12)

$$(4.13) \quad \begin{aligned} (Q_{\mathcal{H}})_{\bar{z}} &= \{\mathcal{L}_q + 2\mathcal{A}\mathcal{L}\}g_{\bar{z}}g_z^2 + \{\mathcal{L}_{\bar{q}} + 2\mathcal{B}\mathcal{L} + \bar{\mathcal{B}}\mathcal{M}\}g_z|g_z|^2 + \{\mathcal{H}_{\bar{q}}(1 + |q|^2)^2|\mathcal{M}|^2\}\bar{g}_z|g_z|^2 \\ &\quad + \{\mathcal{L}\mathcal{H}_{\bar{q}}\bar{\mathcal{M}}(1 + |q|^2)^2\}g_z|g_z|^2 - \{\mathcal{L}\mathcal{H}_{\bar{q}}\bar{\mathcal{M}}(1 + |q|^2)^2\}g_z|g_z|^2 \\ &= g_z^2 \left(\{\mathcal{L}_q + 2\mathcal{A}\mathcal{L}\}g_{\bar{z}} + \{\mathcal{L}_{\bar{q}} + 2\mathcal{B}\mathcal{L} + \bar{\mathcal{B}}\mathcal{M} - \mathcal{L}\mathcal{H}_{\bar{q}}\bar{\mathcal{M}}(1 + |q|^2)^2\}\bar{g}_z \right) \\ &\quad + |g_z|^2(1 + |g|^2)^2 \left(\{H_{\bar{q}}|\mathcal{M}|^2\}\bar{g}_z + \{\mathcal{L}\mathcal{H}_{\bar{q}}\bar{\mathcal{M}}\}g_z \right) \end{aligned}$$

where again the quantities in brackets are evaluated at $g(z)$. Using finally that \mathcal{L} satisfies (4.7) and the definition of $Q_{\mathcal{H}}$ we obtain from (4.13)

$$(4.14) \quad (Q_{\mathcal{H}})_{\bar{z}} = \alpha Q_{\mathcal{H}} + \beta \overline{Q_{\mathcal{H}}},$$

where $\alpha, \beta : \Sigma \rightarrow \bar{\mathbb{C}}$ are given by

$$\alpha = \mathcal{H}_{\bar{q}}(g)\bar{\mathcal{M}}(g)(1 + |g|^2)^2\bar{g}_z, \quad \beta = \left(\frac{\mathcal{L}_q + 2\mathcal{A}\mathcal{L}}{\bar{\mathcal{M}}} \right)(g) \frac{g_z^2}{g_z}.$$

Since $g_z \neq 0$ on Σ , it is clear that $\alpha(z_0), \beta(z_0)$ take values in \mathbb{C} (i.e. they are finite) whenever $g(z_0) \neq \infty$. So, from (4.14) and $Q_{\mathcal{H}} \neq 0$ we have that

$$(4.15) \quad \frac{|(Q_{\mathcal{H}})_{\bar{z}}|}{|Q_{\mathcal{H}}|} \text{ is locally bounded}$$

around every $z_0 \in \Sigma$ with $g(z_0) \neq \infty$.

When $g(z_0) = \infty$ one can easily show from the above formulas that (4.15) also holds, by considering the map $\xi = 1/g$ around z_0 . Thus, (4.15) holds at all points. This condition is well known to imply that $Q_{\mathcal{H}}$ only has isolated zeros of negative index (see [ADT, Jo] for instance). \square

Note that Claims 1, 2 and 4 prove Theorem 4.3. \square

Let us point out here that the proof of Theorem 4.3 above also holds without the assumption that the surface S is compact. Specifically, we have:

COROLLARY 4.4. *Let S be an immersed surface with prescribed mean curvature $\mathcal{H} \in C^1(\mathbb{S}^2)$ in a metric Lie group X , and assume that its Gauss map $G : S \rightarrow \mathcal{U} := G(S) \subset \mathbb{S}^2$ is an orientation preserving diffeomorphism onto its image.*

Then there exists a complex quadratic differential $Q_{\mathcal{H}} dz^2$ defined for any immersed surface $\psi : \Sigma \rightarrow X$ with prescribed mean curvature \mathcal{H} and Gauss map image contained in \mathcal{U} , so that conditions (1), (2) and (3) in Theorem 4.3 hold.

5. Final remarks

5.1. Necessity of strict convexity. Theorem 1.3 is not true in general if we do not assume that there exists a strictly convex sphere with prescribed mean curvature $\mathcal{H} \in C^1(\mathbb{S}^2)$. In fact, the condition of S being *strictly* convex (i.e. of positive curvature at every point) cannot be weakened to S being just convex (i.e. of non-negative curvature), as the following example shows.

EXAMPLE 5.1. Let $C = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, -1 \leq z \leq 1\}$, and let \mathcal{G} be a smooth rotational convex graph $z = z(x, y)$ on $\overline{D} = \{(x, y) : x^2 + y^2 \leq 1\}$ with $z(\partial D) \equiv 1$, so that $S = C \cup \mathcal{G} \cup (-\mathcal{G})$ is a convex (but not strictly convex) sphere in \mathbb{R}^3 .

Now let $C' = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1, -2 \leq z \leq 2\}$, let \mathcal{G}' be the vertical translation of \mathcal{G} so that its horizontal boundary is contained in the plane $z = 2$, and define $S' = C' \cup \mathcal{G}' \cup (-\mathcal{G}')$, which is again a (not strictly) convex sphere in \mathbb{R}^3 .

It is then clear that there is a diffeomorphism $\phi : S \rightarrow S'$ such that, for every $p \in S$:

- (1) The Gauss map of S at p agrees with the Gauss map of S' at $\phi(p)$.
- (2) The principal curvatures of S at p agree with the principal curvatures of S' at $\phi(p)$.

In particular, S and S' are two convex spheres in \mathbb{R}^3 with the same prescribed mean curvature but which do not coincide up to translation in \mathbb{R}^3 .

5.2. Higher order contact with spheres. Let S, S^* be two immersed surfaces of prescribed mean curvature $\mathcal{H} \in C^1(\mathbb{S}^2)$ in a metric Lie group X , and assume that they have a contact point of order $k \geq 1$ at $p \in S \cap S^*$. We assume that the potential function R satisfies $R(\mathcal{H}(q_0), q_0) \neq 0$ where $q_0 \in \bar{\mathbb{C}}$ denotes the common Gauss map image of both S, S^* at p . By reparametrizing both surfaces in a suitable way we may view S, S^* around p as two conformal immersion $\psi, \psi^* : \mathbb{D} \rightarrow X$ with $\psi(0) = \psi^*(0) = p$, whose Gauss maps g, g^* satisfy $g(0) = g^*(0) =: q_0 \in \bar{\mathbb{C}}$, and such that $g_z(0) = (g^*)_z(0) = 1$. Note that these conditions imply by (3.15) that the conformal factors λ, λ^* verify $\lambda(0) = \lambda^*(0)$.

Also, note that the mean curvatures of ψ and ψ^* also coincide at 0. Thus, S, S^* have a contact point of order $k \geq 2$ at p if and only if their respective Hopf differentials Pdz^2, P^*dz^2 satisfy $P(0) = P^*(0)$. It follows from the expressions of λ and P in (3.15) and (3.17) that, in our conditions, $P(0) = P^*(0)$ is equivalent to $\bar{g}_z(0) = (\bar{g}^*)_z(0)$.

Suppose now that S is a compact surface of prescribed mean curvature \mathcal{H} whose Gauss map is a diffeomorphism into $\bar{\mathbb{C}}$. Note that in this case the condition $R(\mathcal{H}(q_0), q_0) \neq 0$ holds automatically for every $q_0 \in \bar{\mathbb{C}}$, see Section 4. Let $Q_{\mathcal{H}} dz^2$ denote the complex quadratic differential associated to S , given by (4.3); note that $Q_{\mathcal{H}} dz^2$ is defined for any conformally immersed surface in X with prescribed mean curvature \mathcal{H} .

Then, using the previous discussion together with the fact that $Q_{\mathcal{H}} \equiv 0$ on S , it is easy to check that a surface S^* of prescribed mean curvature \mathcal{H} in X has a point $p \in S^*$ with $Q_{\mathcal{H}}(p) = 0$ if and only if S^* has a contact of order $k \geq 2$ at p with a left translation of the sphere S .

In addition, as $Q_{\mathcal{H}} dz^2$ only has isolated zeros of negative index on S^* by Theorem 4.3, the Poincaré-Hopf theorem shows that if S^* is compact and of genus $g \geq 1$, then the number of zeros of $Q_{\mathcal{H}} dz^2$ on S^* (counted with multiplicities) is finite and equal to $4g - 4$.

As a consequence of this discussion and the existence of the Guan-Guan spheres in \mathbb{R}^3 for $\mathcal{H} \in C^2(\mathbb{S}^2)$ with $\mathcal{H}(x) = \mathcal{H}(-x) > 0$, we have the next corollary:

COROLLARY 5.2. Let $\mathcal{H} \in C^2(\mathbb{S}^2)$ satisfy $\mathcal{H}(x) = \mathcal{H}(-x) > 0$ for every $x \in \mathbb{S}^2$, let $S_{\mathcal{H}} \subset \mathbb{R}^3$ be the Guan-Guan sphere for \mathcal{H} , and let Σ be a compact immersed surface in \mathbb{R}^3 of genus g with prescribed mean curvature \mathcal{H} . Then, up to a translation in \mathbb{R}^3 :

- (1) If $g = 0$, then $\Sigma = S_{\mathcal{H}}$.
- (2) If $g = 1$, then Σ has no point of contact with $S_{\mathcal{H}}$ of order greater than one.
- (3) If $g \geq 2$, then Σ has at most $4g - 4$ points of contact with $S_{\mathcal{H}}$ of order greater than one.

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JOSÉ A. GÁLVEZ, DEPARTAMENTO DE GEOMETRÍA Y TOPOLOGÍA, UNIVERSIDAD DE GRANADA, 18071 GRANADA, SPAIN

E-mail address: jagalvez@ugr.es

PABLO MIRA, DEPARTAMENTO DE MATEMÁTICA APLICADA Y ESTADÍSTICA, UNIVERSIDAD POLITÉCNICA DE CARTAGENA, 30203 CARTAGENA, MURCIA, SPAIN.

E-mail address: pablo.mira@upct.es